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Completely Unitary Multiparticle Models*

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ABSTRACT

A class of completely unitary multiparticle models is studied. This class is defined by an effective Hamiltonian which allows direct transitions only between two and $n(\geq 2)$ particle states. The non-zero matrix elements however are completely arbitrary. Unitarity-constrained elastic and production amplitudes can be solved exactly and expressed in simple closed forms. It turns out that the absorption functions for the real part of the elastic amplitude, for the imaginary part, and for the production amplitudes are all different. Moreover, alternate black and transparent zones of interaction may develop if the interaction strength is sufficiently strong.

1. INTRODUCTION

A striking feature of high energy reactions is the constancy or the slight rise of total cross section over a wide energy range, a range over which the individual partial cross sections undergo much greater variations. This conspiracy to build up a constant or nearly constant total cross section suggests that unitarity constraints are at work. It is therefore important to understand the nature of these constraints.

The consequences of the unitarity relation having two particles in and two particles out (hereafter called elastic unitarity) are well known. It can be used to prove the Froissart bound[1] or to test various multiparticle models.[2,3] It places a bound in the elastic amplitude at each impact parameter.

In contrast, the consequences of unitarity on production amplitudes are not much known. To the extent that production cross sections are bounded by the total cross section, which through elastic unitarity is related to the elastic amplitude, production amplitudes are already somewhat restricted by elastic unitarity above. Therefore, if the Born amplitude is too large, it must be absorbed in order not to violate this bound. But the exact manner how the absorption should be done is not clear, and there are many different ad hoc ways of doing it.

To make further progress in this direction, presumably we should go beyond elastic unitarity to study the complete unitarity requirements. A completely general study of these is very difficult, so one has to resort to simple models in order to make some headway. There are already several models in the literature[4] where complete unitarity is taken into account.

The class of models we study here is defined by a real effective Hamiltonian with direct transitions only between two and n (≥ 2) particle states. This presumably lacks crossing symmetry but at high energy crossing symmetry in the Hamiltonian may not be all that important. The main advantage of this class of models is the wide range of freedom it allows. The effective Hamiltonian, subject to the above restrictions, is otherwise completely arbitrary. Since only 2- to -n ($n\geq 2$) amplitudes can be measured experimentally, and since all 2- to -n matrix elements of the effective Hamiltonian are arbitrary,[5] there are presumably enough parameters to mimic every conceivable experimental amplitude. Furthermore, this class of models is exactly soluble and the unitarity-constrained amplitude can be expressed in simple closed forms.

Mathematical solutions of the unitary amplitudes and associated cross sections are given in Section 2, while discussions of the results are postponed to Section 3.

2. SCATTERING AMPLITUDES AND CROSS SECTIONS

Let $\mathcal{H}$ be related to the S-matrix by $S = \exp(i\mathcal{H})$. $\mathcal{H}$ is hermitian if S is unitary. We consider in this paper a class of models in which $\mathcal{H}$ is a real and symmetric matrix with non-zero (but arbitrary) matrix elements only between 2- and n-(≥ 2) particle states. Spin is ignored throughout.

The normalizations adopted are as follows. The n-particle states $|k_1 k_2 \ldots k_n>$ with momenta k_i are normalized so that the unit operator is

$$\mathbb{1} = \sum_n \int | k_1 \cdots k_n \rangle \, d\Gamma_n \langle k_1 \cdots k_n | \quad , \tag{1}$$

$$d\Gamma_n = N^{-1} \left(\prod_{i=1}^{n} \frac{d^3 k_i}{(2\pi)^3 2 k_i^0} \right) \quad , \tag{2}$$

where N is unity if all the n-particles are distinct but is otherwise a product of factorials of the number of identical particles. Appropriate summations over hidden indices to ensure a completeness relation in (1) is understood. The T-matrix is normalized so that

$$\langle k_1 \cdots k_n | S | p_1 \cdots p_m \rangle = \langle k_1 \cdots k_n | \mathbb{1} + i (2\pi)^4 \delta^4 (\sum_{j=1}^{n} k_j - \sum_{l=1}^{m} p_l) T | p_1 \cdots p_m \rangle . \tag{3}$$

The n-particle cross section is then

$$\sigma_n = (2s)^{-1} \int | \langle k_1 \cdots k_n | T | p_1 p_2 \rangle |^2 \, d\rho_n(k) \quad , \tag{4}$$

if s is the c. m. energy squared and if the phase space is

$$d\rho_n(k) = d\Gamma_n \, (2\pi)^4 \delta^4 (\sum_{i=1}^{n} k_i - p_1 - p_2) . \tag{5}$$

Similar to (3), we take out an energy-momentum conservation factor from $\mathcal{H}$ and define H by

$$\langle k_1 \cdots k_n | \mathcal{H} | p_1 p_2 \rangle = (2\pi)^4 \delta^4 (\sum_{i=1}^{n} k_i - p_1 - p_2) \langle k_1 \cdots k_n | H | p_1 p_2 \rangle . \tag{6}$$

With the help of the operator

$$R = \sum_n \int |k_1 \cdots k_n\rangle \, d\rho_n(k) \langle k_1 \cdots k_n| \quad , \tag{7}$$

the T-matrix is given in terms of H by

$$T = \mathrm{Re}\,T + i\,\mathrm{Im}\,T \quad , \tag{8}$$

$$\mathrm{Re}\,T = \sum_{\ell=0}^{\infty} \frac{(HR)^{2\ell} H (-1)^{\ell}}{(2\ell+1)!} \quad , \tag{9}$$

$$\mathrm{Im}\,T = \sum_{\ell=0}^{\infty} \frac{(HR)^{2\ell+1} H (-1)^{\ell}}{(2\ell+2)!} \quad . \tag{10}$$

We express the matrix elements of H by their impact-parameter representations. Because of transverse momentum conservation, a matrix element connecting an n-particle state depends only on n-1 independent transverse momenta $\vec{k}_i$, and hence n-1 impact parameters $\vec{b}_i$. If we denote the scaled longitudinal momentum $2k_i^{\ell}/\sqrt{s}$ by x_i, then the most general real matrix element of H can be written in the form

$$\langle k_1 \cdots k_n | H | p_1 p_2 \rangle = \int f_n(\vec{b}_i, x_i, s) \prod_{j=1}^{n-1} \exp(i\,\vec{k}_j \cdot \vec{b}_j)\, d^2 b_j \quad , \tag{11}$$

$$f_n^*(\vec{b}_i, x_i, s) = f_n(-\vec{b}_i, x_i, s) \quad , \tag{12}$$

if the initial transverse momenta are zero. If not, the corresponding element at high energy can be obtained from (11) by a slight rotation to yield

$$\langle k_1 \cdots k_n | H | p_1 p_2 \rangle = \int f_n(\vec{b}_i, x_i, s) \prod_{j=1}^{n-1} \exp\left(i(\vec{k}_j - x_j \vec{p}_1) \cdot \vec{b}_j\right) d^2 b_j \quad . \tag{13}$$

To facilitate computation of the scattering amplitudes, let us divide H into two parts, $H = H' + H_2$, where H_2 has only the 2-to-2 matrix element and H' has only matrix elements connecting 2 to m(>2) particle states. If the initial state has two particles, then it is connected through the operator

$$h'(i) \equiv (H'R)^{i-1} H' \tag{14}$$

only to m-(2-) particle final states if i is odd (even). From (7) and (13), we obtain the matrix element of $h'(2)$ to be

$$\begin{aligned} \langle q_1 q_2 | h'(2) | p_1 p_2 \rangle &= \sum_m \langle q_1 q_2 | H' | k_1 \cdots k_m \rangle d\rho_m(k) \langle k_1 \cdots k_m | H' | p_1 p_2 \rangle \\ &= 2s \int \exp(i \vec{q}_1 \cdot \vec{b}) \, \omega^2(\vec{b}, s) \, d^2 b \quad , \end{aligned} \tag{15}$$

where $\vec{p_1}$ is assumed to be zero and where

$$\omega^2(\vec{b}, s) = \sum_{m=3}^{\infty} \omega_m^2(\vec{b}, s) \quad , \tag{16}$$

$$\omega_m^2(\vec{b}, s) = (2s)^{-1} \int \delta^2\Big(\vec{b} - \sum_{i=1}^{m-1} x_i \vec{b}_i\Big) \big| f_m(\vec{b}_j, x_j, s) \big|^2 \Big(\prod_{k=1}^{m-1} d^2 b_k\Big) d\rho_m^{\ell}(k) \tag{17}$$

The expression $d\rho_m^{\ell}(k)$ stands for the longitudinal part of the phase space factor $d\rho_m(k)$:

$$d\rho_m^{\ell}(k) = \Big(\prod_{i=1}^{m} \frac{dk_i^{\ell}}{(2\pi) 2k_i^0}\Big) (2\pi)^2 \delta\Big(\sum_{i=1}^{m} k_i^{\ell} - (p_1 + p_2)^{\ell}\Big) \delta\Big(\sum_{i=1}^{m} k_i^0 - (p_1 + p_2)^0\Big) . \tag{18}$$

The slight dependence of k_i^0 on $\vec{k}_i$ has been ignored in the computation of (15), an approximation which is certainly valid at large s if x_i is not too small.

Similarly, one can compute various matrix elements of $h'(2\ell)$ for a general ℓ with the help of the formula

$$d\rho_2(k) = \frac{d^2\vec{k}}{(2\pi)^2 2s} \quad . \tag{19}$$

The result is

$$\langle q_1 q_2 | h'(2\ell) | p_1 p_2 \rangle = 2s \int \exp(i\vec{q}_1\cdot\vec{b})\, \omega^{2\ell}(\vec{b},s)\, d^2b \quad , \tag{20}$$

$$\langle q_1 q_2 | h'(2\ell_1) H_2 h'(2\ell_2) H_2 \cdots H_2 h'(2\ell_p) | p_1 p_2 \rangle =$$

$$= 2s \int \exp(i\vec{q}_1\cdot\vec{b})\, \omega^{2\ell}(\vec{b},s)\, y^{p-1}(\vec{b},s)\, d^2b \quad , \tag{21}$$

where

$$y(\vec{b},s) = f_2(\vec{b},s)/2s \quad ,$$

$$\ell = \sum_{i=1}^{p} \ell_i \quad . \tag{22}$$

With the help of these formulas, we may now calculate the matrix elements of

$$h(i) \equiv (HR)^{i-1} H \quad . \tag{23}$$

First we consider the elastic scattering amplitude. If we expand H into a sum of H' and H_2 in the computation of $\langle q_1 q_2 | h(i) | p_1 p_2 \rangle$, then because of the nature of $h'(i)$ mentioned previously, we are allowed to have an even (odd) number of H_2 if i is even (odd). For i even, $i = 2\ell$, the number of ways we can have 2m H_2 inserted into ℓ-m pairs of H' is $\binom{\ell+m}{2m}$. For i odd, $i = 2\ell+1$, the number of ways to have 2m+1 H_2 inserted into ℓ-m pairs of H' is $\binom{\ell+m+1}{2m+1}$. With the help of (9), (10) and (21), we can now write down the elastic scattering amplitude

$$T_{22} = \langle q_1 q_2 | T | p_1 p_2 \rangle = T_{22}^R + i\, T_{22}^I \tag{24}$$

to be

$$T_{22}^I = \sum_{\ell=0}^{\infty} \langle q_1 q_2 | \frac{(-1)^\ell h(2\ell+2)}{(2\ell+2)!} | p_1 p_2 \rangle =$$

$$= 2s \int e^{i\vec{q}_1\cdot\vec{b}}\, d^2b \left\{ 1 + \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell+1}}{(2\ell)!} \sum_{m=0}^{\ell} \binom{\ell+m}{2m} \omega^{2(\ell-m)}(\vec{b},s)\, \nu^{2m}(\vec{b},s) \right\}, \tag{25}$$

$$T_{22}^R = \sum_{\ell=0}^{\infty} \langle q_1 q_2 | \frac{(-1)^\ell h(2\ell+1)}{(2\ell+1)!} | p_1 p_2 \rangle =$$

$$= 2s \int e^{i\vec{q}_1\cdot\vec{b}}\, d^2b \left\{ \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{(2\ell+1)!} \sum_{m=0}^{\ell} \binom{\ell+m+1}{2m+1} \omega^{2(\ell-m)}(\vec{b},s)\, \nu^{2m+1}(\vec{b},s) \right. . \tag{26}$$

Similarly we can calculate the production amplitudes

$$T_{m2} = \langle k_1 \cdots k_m | T | p_1 p_2 \rangle = T^R_{m2} + i\, T^I_{m2} \tag{27}$$

by using (11). The result is

$$\begin{aligned} T^R_{m2} &= \sum_{\ell=0}^{\infty} \langle k_1 \cdots k_m | \frac{h(2\ell+1)(-1)^\ell}{(2\ell+1)!} | p_1 p_2 \rangle = \\ &= \sum_{\ell=0}^{\infty} \langle k_1 \cdots k_m | H' | q_1 q_2 \rangle \, d\rho_2(q) \langle q_1 q_2 | \frac{h(2\ell)(-1)^\ell}{(2\ell+1)!} | p_1 p_2 \rangle = \\ &= \int \Big(\prod_{j=1}^{m-1} d^2 b_j \, e^{i \vec{k}_j \cdot \vec{b}_j} \Big) f_m(\vec{b}_i, x_i, s) \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(2\ell+1)!} \sum_{m=0}^{\ell} \binom{\ell+m}{2m} \omega^{2(\ell-m)}(\vec{B},s)\, \nu^{2m}(\vec{B},s), \end{aligned} \tag{28}$$

$$\begin{aligned} T^I_{m2} &= \sum_{\ell=0}^{\infty} \langle k_1 \cdots k_m | \frac{h(2\ell+2)(-1)^\ell}{(2\ell+2)!} | p_1 p_2 \rangle = \\ &= \sum_{\ell=0}^{\infty} \langle k_1 \cdots k_m | H' | q_1 q_2 \rangle \, d\rho_2(q) \langle q_1 q_2 | \frac{h(2\ell+1)(-1)^\ell}{(2\ell+2)!} | p_1 p_2 \rangle = \\ &= \int \Big(\prod_{j=1}^{m-1} d^2 b_j \, e^{i \vec{k}_j \cdot \vec{b}_j} \Big) f_m(\vec{b}_i, x_i, s) \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(2\ell+2)!} \sum_{m=0}^{\ell} \binom{\ell+m+1}{2m+1} \omega^{2(\ell-m)}(\vec{B},s)\, \nu^{2m+1}(\vec{B}, \end{aligned} \tag{29}$$

with

$$\vec{B} = \sum_{i=1}^{m-1} x_i \vec{b}_i \tag{30}$$

The double sums over m and ℓ in Eqs. (25), (26), (28) and (29) can be evaluated. We shall explain in detail how this is done for (25), and merely write down the results for the others. The double sum

$$F = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell+1}}{(2\ell)!} \sum_{m=0}^{\ell} \binom{\ell+m}{2m} \omega^{2(\ell-m)} \nu^{2m} \tag{31}$$

can be evaluated by noticing that

$$\sum_{\ell=0}^{\infty} \frac{(-1)^{\ell+1}}{(2\ell)!} \omega^{2\ell} = -\cos\omega \; .$$

Adopting the representation

$$\frac{(-1)^{\ell}}{(2\ell)!} \omega^{2\ell} = -\frac{1}{2\pi i} \int_c \frac{\cos z}{z^{2\ell+1}} \omega^{2\ell} \, dz \; , \tag{32}$$

where c is a circle enclosing the origin with a large enough radius to ensure convergence of the infinite sums below, Eq. (31) can be written in the form

$$F = -\frac{1}{2\pi i} \int_c \frac{\cos z}{z} g(z) \, dz \; , \tag{33}$$

where

$$g(z) = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \left(\frac{\omega}{z}\right)^{2\ell} \binom{\ell+m}{2m} \left(\frac{\nu}{\omega}\right)^{2m} . \tag{34}$$

Using instead the summation indices p = ℓ +m and m, (34) can be summed to give

$$g(z) = \sum_{p=0}^{\infty} \frac{1}{2} \left(\frac{\omega}{z}\right)^{2p} \left[\left(1 + \frac{\nu z}{\omega^2}\right)^p + \left(1 - \frac{\nu z}{\omega^2}\right)^p \right]$$

$$= \frac{z^2}{2} \left[(z^2 - \nu z - \omega^2)^{-1} + (z^2 + \nu z - \omega^2)^{-1} \right] . \tag{35}$$

Finally, F may be obtained by substituting (35) into (33) and evaluating the integral by residue calculas. The result is

$$F = -(\omega_+ \cos \omega_+ + \omega_- \cos \omega_-)/(\omega_+ + \omega_-) , \tag{36}$$

where

$$\omega_\pm = [\omega^2 + (\nu/2)^2]^{1/2} \pm \nu/2 . \tag{37}$$

Combining (25) and (36). We finally get

$$T_{22}^{I} = 2s \int e^{i\vec{q}_1 \cdot \vec{b}} d^2 b \left\{ 1 - \frac{\omega_+(\vec{b},s) \cos \omega_+(\vec{b},s) + \omega_-(\vec{b},s) \cos \omega_-(\vec{b},s)}{\omega_+(\vec{b},s) + \omega_-(\vec{b},s)} \right\} . \tag{38}$$

This technique may be applied to other sums, and we obtain in this way

$$T_{22}^{R} = 2s\int e^{i\vec{q}_1\cdot\vec{b}}\, d^2b \quad \frac{\omega_+(\vec{b},s)\sin\omega_+(\vec{b},s) - \omega_-(\vec{b},s)\sin\omega_-(\vec{b},s)}{\omega_+(\vec{b},s)+\omega_-(\vec{b},s)}\,, \tag{39}$$

$$T_{m2}^{R} = \int\Big(\prod_{j=1}^{m-1} d^2b_j\; e^{i\vec{k}_j\cdot\vec{b}_j}\Big)\, f_m(\vec{b}_i,x_i,s)\; \frac{\sin\omega_+(\vec{B},s)+\sin\omega_-(\vec{B},s)}{\omega_+(\vec{B},s)+\omega_-(\vec{B},s)}\,, \tag{40}$$

$$T_{m2}^{I} = \int\Big(\prod_{j=1}^{m-1} d^2b_j\; e^{i\vec{k}_j\cdot\vec{b}_j}\Big)\, f_m(\vec{b}_i,x_i,s)\; \frac{\cos\omega_+(\vec{B},s)-\cos\omega_-(\vec{B},s)}{\omega_+(\vec{B},s)+\omega_-(\vec{B},s)}\,, \tag{41}$$

where $\vec{B}$ is given by (30). The real and imaginary parts of these amplitudes are so simply related that we can just as easily write the complex amplitudes out explicitly. They are

$$T_{22} = 2si\int e^{i\vec{q}_1\cdot\vec{b}}\, d^2b\,\left\{1 - \frac{\omega_+(\vec{b},s)\,e^{i\omega_+(\vec{b},s)} + \omega_-(\vec{b},s)\,e^{-i\omega_-(\vec{b},s)}}{\omega_+(\vec{b},s)+\omega_-(\vec{b},s)}\right\}, \tag{42}$$

$$T_{m2} = i\int\Big(\prod_{j=1}^{m-1} d^2b_j\; e^{i\vec{k}_j\cdot\vec{b}_j}\Big)\, f_m(\vec{b}_i,x_i,s)\; \frac{e^{-i\omega_+(\vec{B},s)} - e^{+i\omega_-(\vec{B},s)}}{\omega_+(\vec{B},s)+\omega_-(\vec{B},s)}\,. \tag{43}$$

From (4), (17), (38)-(43), we can calculate the production cross section $\sigma_m (m > 2)$, the total inelastic cross-section σ, the total cross section σ_T, and the elastic cross section σ_e. The result is

$$\sigma_m = \int d^2b \, \frac{\omega_m^2(\vec{b},s)}{2\left[\omega^2(\vec{b},s)+\nu^2(\vec{b},s)/4\right]} \left\{ 1 - \cos\left[2\left(\omega^2(\vec{b},s)+\nu^2(\vec{b},s)/4\right)^{1/2}\right]\right\}, \tag{44}$$

$$\sigma = \sum_{m=3}^{\infty} \sigma_m = \int d^2b \, \frac{\omega^2(\vec{b},s)}{2\left[\omega^2(\vec{b},s)+\nu^2(\vec{b},s)/4\right]} \left\{ 1 - \cos\left[2\left(\omega^2(\vec{b},s)+\nu^2(\vec{b},s)/4\right)^{1/2}\right]\right\}, \tag{45}$$

$$\sigma_T = 2\int d^2b \left\{ 1 - \frac{\omega_+(\vec{b},s)\cos\omega_+(\vec{b},s) + \omega_-(\vec{b},s)\cos\omega_-(\vec{b},s)}{\omega_+(\vec{b},s)+\omega_-(\vec{b},s)} \right\}, \tag{46}$$

$$\sigma_e = \sigma_T - \sigma \tag{47}$$

We know that the elastic amplitude is almost purely imaginary at high energy. It is therefore instructive to look at the special case when $\nu = 0$. In that limit, the elastic amplitude is purely imaginary and the production amplitude is completely real. We have in fact

$$T_{22} = 2si \int e^{i\vec{q}_1\cdot\vec{b}}\, d^2b \left[\, 1 - \cos\omega(\vec{b},s) \right] \quad , \tag{48}$$

$$T_{m2} = \int \Big(\prod_{j=1}^{m-1} d^2b_j\, e^{i\vec{k}_j\cdot\vec{b}_j} \Big)\, f_m(\vec{b}_i, x_i, s)\, \frac{\sin\omega(\vec{B},s)}{\omega(\vec{B},s)} \quad , \tag{49}$$

$$\sigma_m = \int d^2b \left(\frac{\omega_m(\vec{b},s)}{\omega(\vec{b},s)} \right)^2 \left[\, 1 - \cos^2\omega(\vec{b},s) \right] \quad , \tag{50}$$

$$\sigma = \int d^2b \left[\, 1 - \cos^2\omega(\vec{b},s) \right] \quad , \tag{51}$$

$$\sigma_T = 2 \int d^2b \left[\, 1 - \cos\omega(\vec{b},s) \right] \quad , \tag{52}$$

$$\sigma_e = \int d^2b \left[\, 1 - \cos\omega(\vec{b},s) \right]^2 \quad . \tag{53}$$

The meaning of these equations will be discussed in the next Section.

3. DISCUSSIONS

In view of the smallness of the real part of the elastic scattering amplitude at high energies, we shall concentrate most of our discussions in the $\nu = 0$ limit. The opaqueness $\Omega(\vec{b}, s)$, defined by[6]

$$T_{22} = 2s\, i \int e^{i\vec{q}_1 \cdot \vec{b}}\, d^2b \left[1 - e^{-\Omega(\vec{b},s)} \right] , \qquad (54)$$

is given by (48) to be

$$\Omega(\vec{b},s) = -\ln\left[\cos\omega(\vec{b},s)\right] . \qquad (55)$$

Thus the real part of Ω is always positive, which is a well known consequence of elastic unitarity. So are Eqs. (51) - (53), and we shall therefore not discuss them any further.

In the weak interaction limit, $\omega << 1$, (55) becomes

$$\Omega(\vec{b},s) = \omega^2(\vec{b},s)/2 \ , \quad (\omega << 1) , \qquad (56)$$

which according to (16) receives additive contributions from different channels. When the interaction is not weak, although different channels still contribute additively to ω^2, they no longer do so to Ω. This is a consequence of the continuous creation and absorption of multiparticle states dictated by the unitarity requirements in the model.

To see what unitarity does to multiparticle cross sections, let us first look at Eq. (50). We can extract from it the internal multiplicity distribution function[7] P_m, which measures the probability density of finding m particles at impact parameter $\vec{b}$:

$$P_m = \omega_m^2(\vec{b},s) / \omega^2(\vec{b},s) \quad . \tag{57}$$

This expression is easy to understand in the weak coupling limit, when (See (50) and (51)) $\omega_m^2(\vec{b},s)/2$ is the cross section density for producing m particles at impact parameter $\vec{b}$, and $\omega^2(\vec{b},s)$ is the total inelastic cross section density there. Their ratio is therefore the probability density, as indicated by (57). What Eq. (50) goes beyond this to say is that no matter how strong the interaction is, the ratio on the right hand side of (57) is still the probability density. This simplification is a result of the assumption that direct m -to n-particle conversions for m> 2 are absent in the interactions.

Going beyond multiplicity distributions, we see from (11) and (49) that the m-particle amplitude differs from its Born term only by the multiparticle absorption factor $\sin\omega(\vec{B},s)/\omega(\vec{B},s)$. There are two things to notice here. The effective impact parameter $\vec{B}$, given by (30), is determined simply by angular momentum conservation. It is therefore a general feature independent of our specific model assumptions. Secondly, the absorption function $\sin\omega/\omega$ here is different from the absorption function $\cos\omega$ in the imaginary part of the elastic amplitude, Eq. (48). It is true that they both have the same general character, that they are bounded by 1 and tend to unity as $\omega \to 0$. Moreover, if $0 \leq \omega \leq \pi/2$, they both decrease monotonically as ω increases, with $\sin\omega/\omega$ always larger than $\cos\omega$. But beyond $\pi/2$, both of them oscillate out of phase with each other.

It is also interesting to note that the real part of the elastic scattering amplitude has an absorption function that is the arithmetic mean of the two: $(\cos\omega + \sin\omega/\omega)/2$. This can be obtained by expanding Eq. (30) to first power in ν.

Finally, let us consider in more detail the behavior of opaqueness. If f_m vanishes at large impact parameter, so will ω (Eqs.(16) and (17)), and therefoer Ω (Eq. (55)). This is to be expected. When the impact parameter decreases from infinity, Ω increases and the interaction region becomes more and more opaque. If the interaction is not strong enough to make ω exceed $\pi/2$ anywhere, then Ω is a positive monotomic function of ω, and the interaction region gets more opaque wherever the effective interaction is stronger. This corresponds to the usual picture of geometrical models. However, if the interaction becomes very strong and exceeds $\pi/2$, then although $\omega(\vec{b}, s)$ may still be a monotonic function of b, $\Omega(\vec{b}, s)$ will develop alternate black and transparant zones because of (55). Such a zebra-like structure is very interesting because it means that in principle there is an impact parameter (e.g., $\omega = 2\pi$) within the overall interaction range that we can aim at where no interaction whatsoever occurs. At present energies, presumably the interaction is not strong enough for zebra zones to occur. But with the increase of total cross section and thereby the indication of interactions getting stronger for larger energies, it is not completely ruled out that such zebra zones may develop at ultra high energies.

In closing, we should also mention that if the real part of the elastic scattering amplitude is taken into account, $\nu \neq 0$, one changes the quantitative but not the qualitative features discussed above. For example, Eq. (55) would be slightly modified so that a completely black interaction region ($\Omega = \infty$) can no longer occur.

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REFERENCES AND FOOTNOTES

[1] M. Froissart, Phys. Rev. 123, 1053 (1961).

[2] R.C. Hwa, Univ. of Oregon preprint, to appear in Phys.Rev.

[3] C.S. Lam and R.T. Sharp, McGill University preprint, to appear in Phys. Rev.

[4] S. Auerbach, R. Aviv, R. Sugar and R. Blankenbeckler, Phys. Rev. Letters 29, 522 (1972); Phys. Rev. D6, 2216 (1972); G. Calucci, R. Jengo and C. Rebbi, Nuovo Cimento 4A, 330 (1971).

[5] In the text, we assume for the sake of clearness the effective Hamiltonian to be real. The extension to a complex case is trivial.

[6] T.T. Chou and C.N. Yang, Phys. Rev. 170, 1591 (1968).

[7] C.S. Lam, McGill University preprint.